

# Comment on “A new, exact, gauge-invariant RG-flow equation”

Filipe Paccetti Correia<sup>1</sup>

*Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16,  
D-69120 Heidelberg, Germany*

## Abstract

We show that the exact RG-flow equation introduced recently in hep-th/0207134 can be obtained in the sharp cut-off limit of the well-known ERGE. This can be expected from the fact that in this limit the new scale-dependent effective action coincides with the one which is usually considered.

---

<sup>1</sup>E-mail address: F.Paccetti@ThPhys.Uni-Heidelberg.DE

The purpose of this short note is to show that the *new* exact RG-flow equation recently proposed in ref.[1], which is obtained by a partial Legendre transform, corresponds to the *sharp* cut-off limit of the *exact* RG equation (ERGE) of ref.[2, 3]<sup>2</sup>. To do this we will start by deriving the ERGE before considering the sharp cut-off limit.

**The effective action.** The scale-dependent 1PI effective action can be defined in the following way: One introduces the quadratic cut-off functional

$$\mathcal{O}_k[\chi - \varphi] \equiv \exp(-\Delta_k S[\chi - \varphi]) \equiv \exp\left(-\frac{1}{2} \int (\chi - \varphi) R_k (\chi - \varphi)\right), \quad (1)$$

inside the functional integral which defines the partition function,

$$e^{W[J]} \equiv \int \mathcal{D}\chi e^{-S[\chi] + \int J\chi}. \quad (2)$$

We assume that  $\lim_{k \rightarrow 0} R_k = 0$  and therefore the functional

$$e^{\widehat{W}_k[J, \varphi]} \equiv \int \mathcal{D}\chi \mathcal{O}_k[\chi - \varphi] e^{-S[\chi] + \int J\chi}, \quad (3)$$

obtained with this procedure converges to  $W[J]$  as  $k \rightarrow 0$ . We can now perform a Legendre transformation of  $\widehat{W}_k$  with respect to  $J$  to obtain

$$\widehat{\Gamma}_k[\phi, \varphi] = -\widehat{W}_k[J, \varphi] + \int J\phi, \quad \phi \equiv \frac{\delta \widehat{W}_k[J, \varphi]}{\delta J}. \quad (4)$$

This means that  $\widehat{\Gamma}_k$  is given implicitly by

$$\exp\left(-\widehat{\Gamma}_k[\phi, \varphi]\right) = \int \mathcal{D}\chi \mathcal{O}_k[\chi - \varphi + \phi] \exp\left(-S[\chi + \phi] + \int \frac{\delta \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi} \chi\right). \quad (5)$$

Since it is our intention to obtain an effective action which interpolates between the classical action  $S[\phi]$  in the *UV* ( $k \rightarrow \infty$ ), and the full effective action  $\Gamma[\phi]$  in the *IR* ( $k = 0$ ) we first assign  $\mathcal{O}_k[\chi - \varphi]$  the following property

$$\lim_{k \rightarrow \infty} \mathcal{O}_k[\chi] \sim \delta[\chi], \quad (6)$$

where the r.h.s. is a  $\delta$ -functional. This can be obtained if  $\lim_{k \rightarrow \infty} R_k = \infty$ . In this case we have

$$\lim_{k \rightarrow \infty} \widehat{\Gamma}_k[\phi, \varphi] = S[\varphi] + \int \frac{\delta \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi} (\phi - \varphi). \quad (7)$$

---

<sup>2</sup> For a comprehensive review see [4]

Finally, to obtain the desired property, we set  $\varphi = \phi$ :

$$\Gamma_k[\phi] \equiv \widehat{\Gamma}_k[\phi, \phi]. \quad (8)$$

One can use now

$$\frac{\delta \widehat{\Gamma}_k[\phi, \varphi]}{\delta \varphi} = -\frac{\delta \widehat{W}_k[J, \varphi]}{\delta \varphi} = R_k(\varphi - \phi), \quad (9)$$

to write  $\Gamma_k[\phi]$  as (see eq.(5))

$$\exp(-\Gamma_k[\phi]) = \int \mathcal{D}\chi \mathcal{O}_k[\chi] \exp\left(-S[\chi + \phi] + \int \frac{\delta \Gamma_k[\phi]}{\delta \phi} \chi\right). \quad (10)$$

From this expression it is not difficult to recognize  $\Gamma_k[\phi]$  as being given (perturbatively) by the sum of the connected 1PI vacuum graphs with  $\phi$ -dependent vertices and internal lines regulated by the introduced cut-off.

The attentive reader may consider the above formalism reminiscent of the background field method used to ensure the gauge invariance of the 1PI effective action (see [5] and references therein). And indeed both ideas can be combined to define a scale-dependent, gauge invariant, 1PI effective action [6].

**The flow equation.** To calculate the exact renormalization group equation (ERGE) for  $\Gamma_k$  let us note that

$$\begin{aligned} \partial_k \Gamma_k[\phi] &= \partial_k \widehat{\Gamma}_k[\phi, \phi] = -\partial_k \widehat{W}_k[J, \phi] = \frac{1}{2} \int \partial_k R_k \langle (\chi - \phi)(\chi - \phi) \rangle \\ &= \frac{1}{2} \int \partial_k R_k \frac{\delta^2 \widehat{W}_k[J, \phi]}{\delta J \delta J}. \end{aligned} \quad (11)$$

Now, we have

$$\frac{\delta^2 \widehat{W}_k[J, \phi]}{\delta J \delta J} = \left( \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \phi} \Big|_{\varphi=\phi} \right)^{-1}, \quad (12)$$

while

$$\frac{\delta^2 \Gamma_k[\phi]}{\delta \phi \delta \phi} = \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \phi} \Big|_{\varphi=\phi} + 2 \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \varphi} \Big|_{\varphi=\phi} + \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \varphi \delta \varphi} \Big|_{\varphi=\phi}. \quad (13)$$

Using eq.(9) one gets

$$\frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \phi} \Big|_{\varphi=\phi} = \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi \delta \phi} + R_k, \quad (14)$$

obtaining in this way the well-known ERGE for  $\Gamma_k$  [2, 3]:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int (\partial_k R_k) \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}. \quad (15)$$

**Sharp cut-off limit.** By sharp cut-off one means a function  $R_k$  which diverges for momenta below the scale  $k$  and vanishes above this scale. Although one could think that in this limiting case  $\Gamma_k$  is not well defined due to an ill-definition of the Legendre transform of  $W_k[J, \varphi]$  we will see that this is not the case.

In this limit  $\mathcal{O}_k$  is of the form

$$\mathcal{O}_k[\chi] \sim \prod_{p^2 < k^2} \delta(\chi(p)), \quad (16)$$

which means that (using the notation of ref.[1]) we can write

$$\exp W_k[J, \varphi_0^k] = \int \mathcal{D}\chi_k^\Lambda \exp \left( -S[\varphi_0^k + \chi_k^\Lambda] + \int J(\varphi_0^k + \chi_k^\Lambda) \right), \quad (17)$$

where  $\varphi_0^k$  contains only Fourier modes with  $p < k$  and  $\chi_k^\Lambda$  only modes with  $p > k$ . This differs from the  $W_k$  ( $\equiv \mathcal{W}_k$ ) defined in [1] in the following way<sup>3</sup>

$$W_k[J, \varphi_0^k] = \mathcal{W}_k[J_k^\Lambda, \varphi_0^k] + \int J_0^k \varphi_0^k, \quad (18)$$

but, as we will show, the scale dependent effective action  $\Gamma_k$  coincides with the one defined in that work (we will call this one  $\mathcal{G}_k$ ). This follows from the fact that  $\Gamma_k[\phi] = \widehat{\Gamma}_k[\phi, \phi]$  in this case is given by

$$\Gamma_k[\phi] = \widehat{\Gamma}_k[\phi, \varphi_0^k], \quad (19)$$

since due to the sharp cut-off we have  $\phi = \varphi_0^k$  for  $p < k$  and  $W_k$  is independent of  $\varphi_k^\Lambda$ , where  $\varphi_k^\Lambda = \varphi$  for  $p > k$ . Furthermore, we have

$$\Gamma_k[\phi] = -W_k[J, \varphi_0^k] + \int J(\varphi_0^k + \phi_k^\Lambda) = -\mathcal{W}_k[J_k^\Lambda, \varphi_0^k] + \int J_k^\Lambda \phi_k^\Lambda, \quad (20)$$

where

$$\phi_k^\Lambda = \frac{\delta \mathcal{W}_k[J_k^\Lambda, \varphi_0^k]}{\delta J_k^\Lambda}. \quad (21)$$

But this is clearly the definition of  $\mathcal{G}_k$ , the Legendre transform of  $\mathcal{W}_k$  with respect to  $J_k^\Lambda$ , as we intended to show.

---

<sup>3</sup> There is also a irrelevant difference in sign.

**The flow equation in the sharp cut-off limit.** Let us now see what happens to the flow equation (15) in the sharp cut-off limit. The flow equation can be rewritten as

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{tr} \tilde{\partial}_k \ln (\Gamma_k^{(2)}[\phi] + R_k) = \frac{1}{2} \tilde{\partial}_k \ln \det (\Gamma_k^{(2)}[\phi] + R_k), \quad (22)$$

where  $\tilde{\partial}_k$  only acts upon  $R_k$ . Since below  $k$  the cut-off function  $R_k$  diverges we get

$$\partial_k \Gamma_k[\phi] = \lim_{\delta k \rightarrow 0} \frac{1}{2\delta k} \left[ \ln \det \frac{\delta^2 \Gamma_k}{\delta \phi_k^\Lambda \delta \phi_k^\Lambda} - \ln \det \frac{\delta^2 \Gamma_k}{\delta \phi_{k-\delta k}^\Lambda \delta \phi_{k-\delta k}^\Lambda} \right], \quad (23)$$

where the  $k$  in the first term and the  $k - \delta k$  in the second one denote the IR cut-offs in the respective determinants and we neglected a  $\phi$ -independent piece. It is not difficult to recognize that this is the flow equation which was obtained by the authors of ref.[1] by other means.

Thus in this paper we showed that the scale-dependent 1PI effective action defined in [1] by a partial Legendre transformation can be considered to be the sharp cut-off limit of the one defined in [2, 3]. As expected, the flow equation of [1] could also be obtained as the same limit of the ERGE of [2, 3].

### Acknowledgements

The author wishes to thank T.Baier, M.G.Schmidt and Z.Tavartkiladze for useful discussions. This work is supported by Fundação de Ciência e Tecnologia (grant SFRH/BD/4973/2001).

## References

- [1] V.Branchina, K.A.Meissner, G.Veneziano, hep-th/0207134.
- [2] C.Wetterich, Phys.Lett. **B301**, (1993) 90.
- [3] M.Bonini, M.D'Attanasio, G.Marchesini, Nucl.Phys. **B409**, (1993) 441; T.R.Morris, Int.J.Mod.Phys. **A9**, (1994) 2411.
- [4] J.Berges, N.Tetradis, C.Wetterich, Phys.Rep. **363**, (2002) 223.
- [5] L.F.Abbot, Nucl.Phys. **B185**, (1981) 189.
- [6] M.Reuter, C.Wetterich, Nucl.Phys. **B417**, (1994) 181.